Probabilistic Bijections for Non-Attacking Fillings

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How to combinatorially prove that generating functions are equal?

Weight-preserving bijection: function $f\colon \mathcal{T} \to \mathcal{U}$ with an inverse $g\colon \mathcal{U} \to \mathcal{T}$ such that, whenever $f(T) = U$ implies $\mathrm{wt}(T) = \mathrm{wt}(U)$. A weight-preserving bijection between $\mathcal{T}$ and $\mathcal{U}$ implies

$$\textstyle\sum_{T\in\mathcal{T}} \mathrm{wt}(T) = \sum_{U\in\mathcal{U}} \mathrm{wt}(U).$$

What if it's impossible to find a weight-preserving bijection?

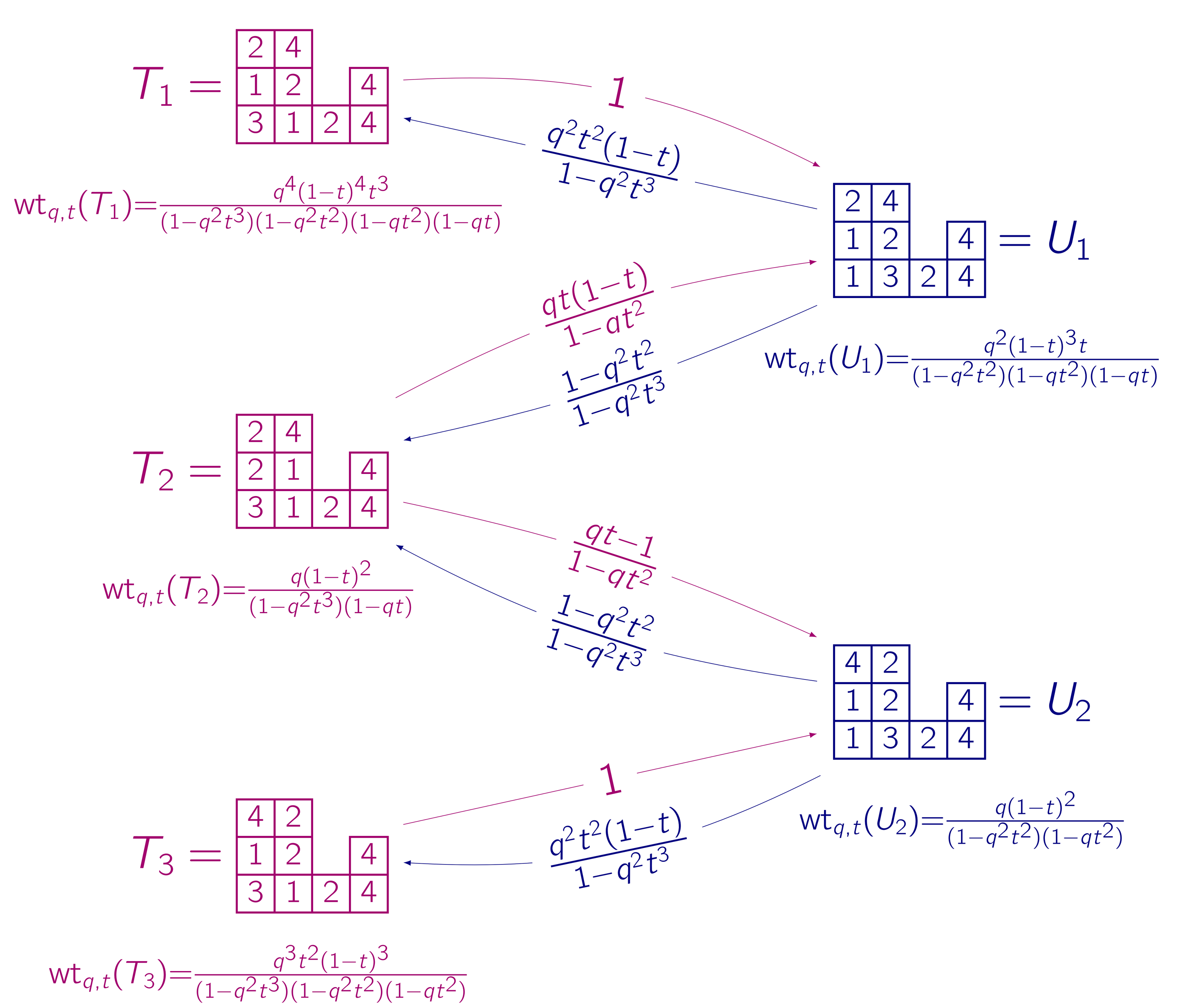
Probabilistic bijection: pair of functions $\mathrm{prob}\colon \mathcal{T} \times \mathcal{U} \to A$ and $\mathrm{prob}'\colon \mathcal{U} \times \mathcal{T} \to A$ such that, for all $T \in \mathcal{T}$ and $U \in \mathcal{U}$,

- $\sum_{U\in\mathcal{U}} \mathrm{prob}(T,U) = 1$, $\sum_{T\in\mathcal{T}} \mathrm{prob}'(U,T) = 1$, and
- $\mathrm{wt}(T)\,\mathrm{prob}(T,U) = \mathrm{wt}(U)\,\mathrm{prob}'(U,T)$. *(balance condition)*

Same conclusion! A probabilistic bijection between $\mathcal{T}$ and $\mathcal{U}$ implies

$$\textstyle\sum_{T\in\mathcal{T}} \mathrm{wt}(T) = \sum_{U\in\mathcal{U}} \mathrm{wt}(U).$$

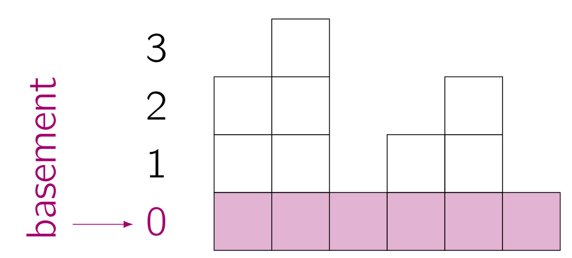
Example & Spoiler



Non-attacking fillings (NAFs)

Augmented skyline diagram of a composition α:

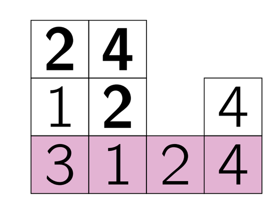
$\alpha = (2\ 3\ 0\ 1\ 2\ 0)$



A pair of boxes is **attacking** if:

- same row
- or
- consecutive rows, top box to the right

A **non-attacking filling** of shape α is a map $T\colon \mathrm{dg}(\alpha) \to [n]$ such that attacking boxes have different entries. The content counts the instances of each i in the filling. The basement of a filling is the permutation $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n] \in S_n$ read from the basement.

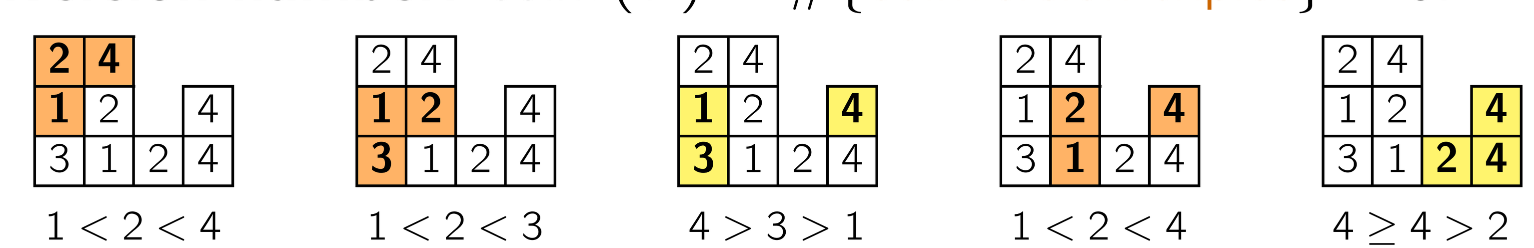


shape $\alpha = (2,2,0,1)$
basement $\sigma = [3,1,2,4]$
content $\beta = (1,2,0,2)$

$x^T = x_1^1 x_2^2 x_3^0 x_4^2$

Major index: $\mathrm{maj}(T) = \sum_{T(u)>T(\text{south of } u)} (1 + \mathrm{leg}(u))$, where $\mathrm{leg}(u)$ is the number of boxes above u. In the example, $\mathrm{maj}(T) = 1 + 1 + 2 = 4$.

Coinversion number: $\mathrm{coinv}(T) = \#\{\text{coinversion triples}\} = 3$.



Weight of a non-attacking filling:

$$\mathrm{wt}_{q,t}(T) = q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \prod_{T(u)\neq T(\text{south of } u)} \frac{1-t}{1-q^{1+\mathrm{leg}(u)}t^{1+\mathrm{arm}(u)}}$$

Example:

$$\mathrm{wt}_{q,t}(T) = q^4t^3 \frac{(1-t)^4}{(1-q^2t^3)(1-q^2t^2)(1-qt^2)(1-qt)}.$$

Generating function formula for E_α^σ [Fer11]

The permuted basement Macdonald polynomial is

$$E_\alpha^\sigma(x_1, x_2, \ldots, x_n; q, t) = \textstyle\sum_{T\in\mathrm{NAF}(\alpha,\sigma)} x^T\, \mathrm{wt}_{q,t}(T).$$

Permuted basement Macdonald polynomials E_α^σ can also be described as eigenfunctions of modified Cherednik–Dunkl operators, or as transformations of nonsymmetric Macdonald polynomials E_α via Demazure–Lusztig operators. Nonsymmetric Macdonald polynomials E_α generalize Demazure characters and atoms, and help understand symmetric Macdonald polynomials P_λ, which extend Jack, Hall–Littlewood, q-Whittaker, and Schur polynomials.

Probabilistic bijection on NAFs

Theorem (D–Mandelshtam, 2025⁺)

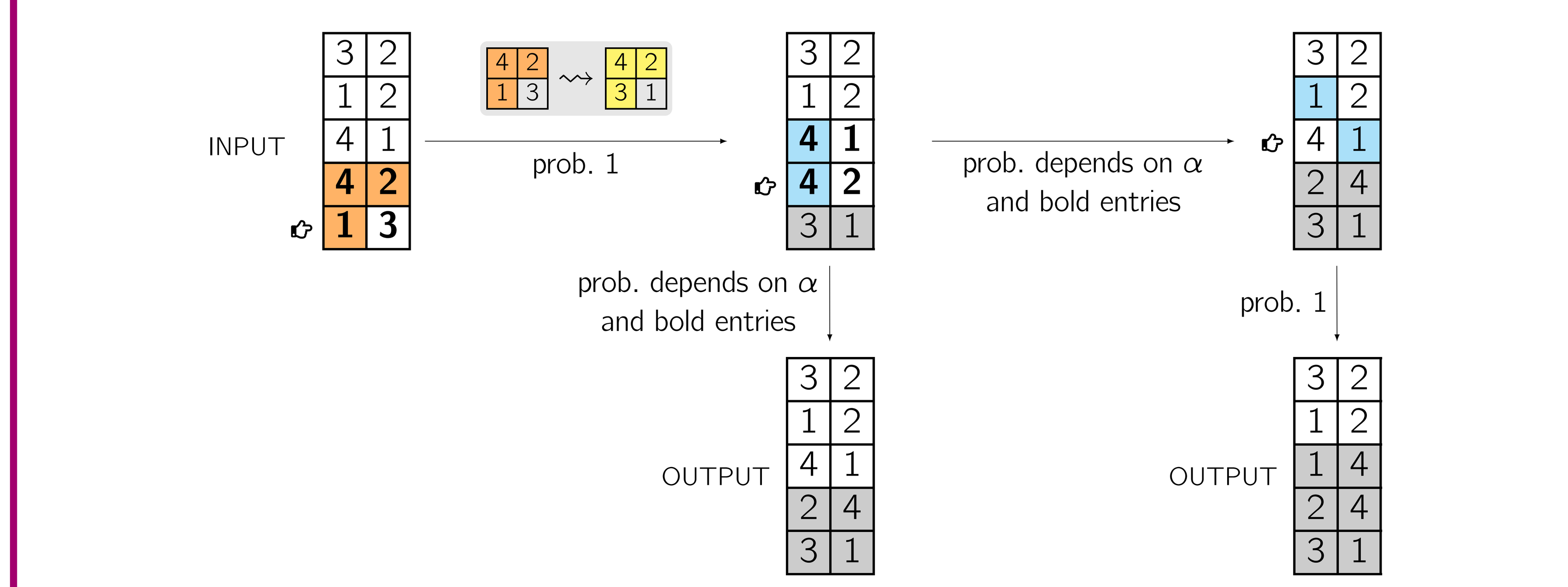
Let α be a composition with $\alpha_i = \alpha_{i+1}$ and σ be a permutation. Then,

$$E_\alpha^\sigma = E_\alpha^{\sigma s_i},$$

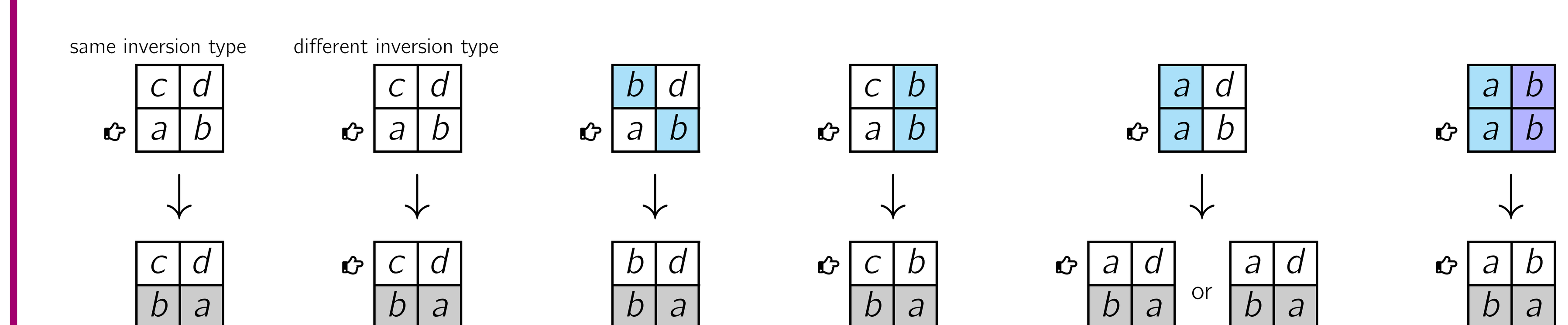
where $\sigma s_i = [\sigma_1, \ldots, \sigma_{i+1}, \sigma_i, \ldots, \sigma_n]$.

Strategy: Construct a probabilistic bijection between $\mathrm{NAF}(\alpha,\sigma)$ and $\mathrm{NAF}(\alpha,\sigma s_i)$ when $\alpha_i = \alpha_{i+1}$.

Definition via example



Rules on moving the pointer (☞)



References

[Fer11] Jeffrey Paul Ferreira. "Row-strict quasisymmetric Schur functions, characterizations of Demazure atoms, and permuted basement nonsymmetric Macdonald polynomials". PhD thesis. California, United States: University of California, Davis, 2011. 90 pp.

[FS24] Gabriel Frieden and Florian Schreier-Aigner. *qtRSK*: A probabilistic dual RSK correspondence for Macdonald polynomials.* 2024. arXiv: 2403.16243.

[Man24] Olya Mandelshtam. *A compact formula for the symmetric Macdonald polynomials.* 2024. arXiv: 2401.17223 [math.CO].