Probabilistic Bijections for Non-Attacking Fillings

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Part I

Macdonald Polynomials and Non-Attacking Fillings

Macdonald Polynomials

Symmetric Macdonald polynomials (Macdonald 1988)

$$P_\lambda(x_1, \ldots, x_n; q, t)$$

where λ is a partition.

- Unique basis under triangularity and orthogonality conditions.
- Extension of Jack polynomials, Hall–Littlewood polynomials, q-Wittaker polynomials, Schur polynomials.

Nonsymmetric Macdonald polynomials (Macdonald 1996)

$$E_\alpha(x_1, \ldots, x_n; q, t)$$

where α is a composition.

- Help understand symmetric Macdonald polynomials.
- Extend Demazure characters (key polynomials) and atom polynomials.

Permuted Basement Macdonald Polynomials

Permuted basement Macdonald polynomials (Ferreira 2011)

$$E_\alpha^\sigma(x_1, \ldots, x_n; q, t)$$

where α is a composition and σ is a permutation.

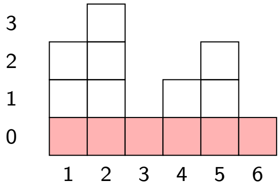
They can be described as:

- a transformation of nonsymmetric Macdonald polynomials by Demazure–Lusztig operators,
- eigenfunctions of a version of Cherednik–Dunkl operators,
- the weight generating functions of **non-attacking fillings**.

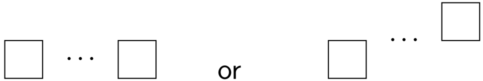
Augmented Skyline Diagrams

Augmented skyline diagram of a composition α: (basement = row 0 = red)

$$\alpha = (2\ \ 3\ \ 0\ \ 1\ \ 2\ \ 0)$$



A pair of boxes is **attacking** if:



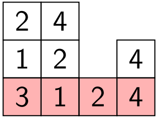
same row — or — consecutive rows, top box to the right

Non-Attacking Fillings

A **non-attacking filling** of shape α is an assignment of numbers $\{1, \dots, n\}$ to the diagram of α such that attacking boxes have different entries.

The **basement** of a filling is the permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in S_n$ of the entries in the basement row, from left to right.

e.g. shape $\alpha = (2, 2, 0, 1)$ and basement $\sigma = [3, 1, 2, 4]$

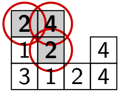


The **content** counts how many i's are in the filling (excluding the basement).

e.g. the filling above has content $(1, 2, 0, 2)$

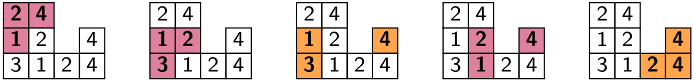
$$x^T = x_1^1 x_2^2 x_3^0 x_4^2$$

- major index: $\mathrm{maj}(T) = \sum_{T(u) > T(\text{south of } u)} (1 + \mathrm{leg}(u)),$



$$\mathrm{maj}(T) = 1 + 1 + 2 = 4$$

- coinversion number: $\mathrm{coinv}(T) = \#\{\text{coinversion triples}\}$



$1 < 2 < 4$ $\quad 1 < 2 < 3$ $\quad 4 > 3 > 1$ $\quad 1 < 2 < 4$ $\quad 4 \geq 4 > 2$

$$\mathrm{coinv}(T) = 3$$

Weight of a Non-Attacking Filling

$$\mathrm{wt}_{q,t}(T) = q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} \prod_{T(u)\neq T(\text{south of } u)} \frac{1-t}{1-q^{1+\mathrm{leg}(u)}t^{1+\mathrm{arm}(u)}}$$

2	4		
1	2		4
3	1	2	4

$$\mathrm{wt}_{q,t}(T) = q^4 t^3 \frac{(1-t)^4}{(1-q^2t^3)(1-q^2t^2)(1-qt^2)(1-qt)}.$$

Generating Function Formula

Generating function formula for E_α^σ (Ferreira 2011)

$$E_\alpha^\sigma = \sum_{T \in \mathrm{NAF}(\alpha,\sigma)} x^T \mathrm{wt}_{q,t}(T).$$

Motivation

Alexandersson (2019) studies symmetries of permuted basement Macdonald polynomials.

- One of the results involves E_λ^σ where σ is the **shortest** permutation that sends λ to a rearrangement α.

Corteel, Mandelshtam, and Williams (2022) study the asymmetric simple exclusion process (ASEP), a model of interacting particles.

- One of the results involves E_λ^τ where τ is the **longest** permutation that sends λ to a rearrangement α.

Theorem (D–Mandelshtam, 2025$^+$)

$$E_\lambda^\sigma = E_\lambda^\tau$$

where σ (resp. τ) is the shortest (resp. longest) permutation that sends λ to a rearrangement α.

Main Result

More generally, we have the following result.

Theorem (D–Mandelshtam, 2025$^+$)

Let α be a composition with $\alpha_i = \alpha_{i+1}$ and σ be a permutation. Then,

$$E_\alpha^\sigma = E_\alpha^{\sigma s_i},$$

where $\sigma s_i = [\sigma_1, \ldots, \sigma_{i+1}, \sigma_i, \ldots, \sigma_n]$.

How to prove it? **Construct a probabilistic bijection** between $\mathrm{NAF}(\alpha, \sigma)$ and $\mathrm{NAF}(\alpha, \sigma s_i)$ when $\alpha_i = \alpha_{i+1}$.

Part II

Probabilistic Bijections

Generalizing the goal: $F = G$

F and G are weight generating functions of **S** and **T**. How to combinatorially prove

$$F = G\ ?$$

A strategy

Construct a **weight-preserving bijection** between **S** and **T**.

Weight-preserving Bijections

Weight-preserving bijections:

$$f : \mathbf{S} \to \mathbf{T} \qquad \text{with an inverse} \qquad g : \mathbf{T} \to \mathbf{S}$$

such that, whenever $f(S) = T$, we have

$$\mathrm{wt}(S) = \mathrm{wt}(T).$$

Proposition

If there exists a weight-preserving bijection between $\mathbf{S}$ *and* $\mathbf{T}$, *then*

$$\sum_{S \in \mathbf{S}} \mathrm{wt}(S) = \sum_{T \in \mathbf{T}} \mathrm{wt}(T).$$

What about when it's hard (or impossible) to find a weight-preserving bijection?

Probabilistic Bijections

Probabilistic bijection: pair of maps to an algebra A (Frieden and Schreier-Aigner 2024)

$$\mathrm{prob}_{\mathbf{S}} : \mathbf{S} \times \mathbf{T} \to A \quad \text{and} \quad \mathrm{prob}_{\mathbf{T}} : \mathbf{T} \times \mathbf{S} \to A$$

such that, for all $S \in \mathbf{S}$ and $T \in \mathbf{T}$,

- **probabilities sum to 1:** $\sum_{T \in \mathbf{T}} \mathrm{prob}_{\mathbf{S}}(S, T) = 1$ and $\sum_{S \in \mathbf{S}} \mathrm{prob}_{\mathbf{T}}(T, S) = 1$,
- **balance condition:** $\mathrm{wt}(S)\,\mathrm{prob}_{\mathbf{S}}(S, T) = \mathrm{wt}(T)\,\mathrm{prob}_{\mathbf{T}}(T, S)$.

Proposition

If there exists a probabilistic bijection between $\mathbf{S}$ *and* $\mathbf{T}$*, then*

$$\sum_{S \in \mathbf{S}} \mathrm{wt}(S) = \sum_{T \in \mathbf{T}} \mathrm{wt}(T).$$

Proof of Proposition

Proposition

If there exists a probabilistic bijection between $\mathbf{S}$ *and* $\mathbf{T}$, *then*

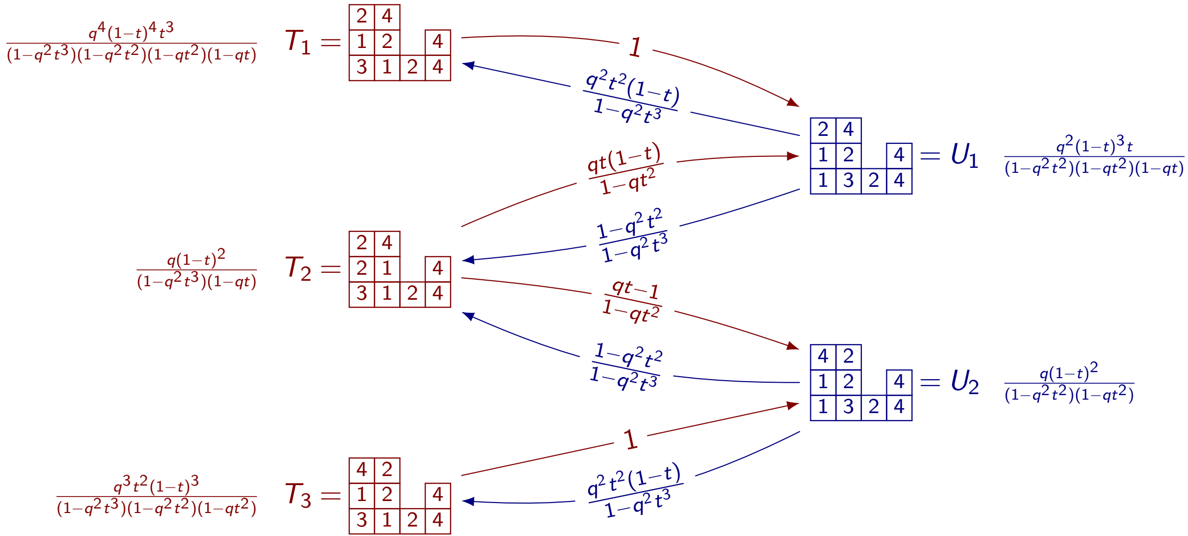
$$\sum_{S \in \mathbf{S}} \mathrm{wt}(S) = \sum_{T \in \mathbf{T}} \mathrm{wt}(T).$$

Proof

$$\begin{aligned}
\sum_{S \in \mathbf{S}} 1 \cdot \mathrm{wt}(S) &= \sum_{S \in \mathbf{S}} \sum_{T \in \mathbf{T}} \mathrm{prob}_{\mathbf{S}}(S, T)\,\mathrm{wt}(S) && \text{(probabilities sum to 1)} \\
&= \sum_{T \in \mathbf{T}} \sum_{S \in \mathbf{S}} \mathrm{prob}_{\mathbf{T}}(T, S)\,\mathrm{wt}(T) && \text{(balance condition)} \\
&= \sum_{T \in \mathbf{T}} 1 \cdot \mathrm{wt}(T). && \text{(probabilities sum to 1)}
\end{aligned}$$

□

Example (Spoiler)



Part III

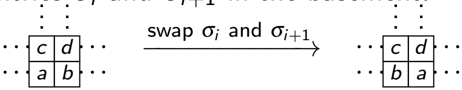
Probabilistic Bijections for Non-Attacking Fillings

Goal and Sketch

Goal

Construct a probabilistic bijection between $\mathrm{NAF}(\alpha, \sigma)$ and $\mathrm{NAF}(\alpha, \sigma s_i)$, where $\alpha_i = \alpha_{i+1}$.

- **Naive idea:** Just swap the entries σ_i and σ_{i+1} in the basement.



- **An issue:** The resulting filling may be attacking.
- **Idea for fix:** If the resulting filling is attacking, swap entries in the top row as well, and so on.
- **Refinement:** To take care of the weights, assign probabilities to each step.

Definition of the Probabilistic Algorithm by Example

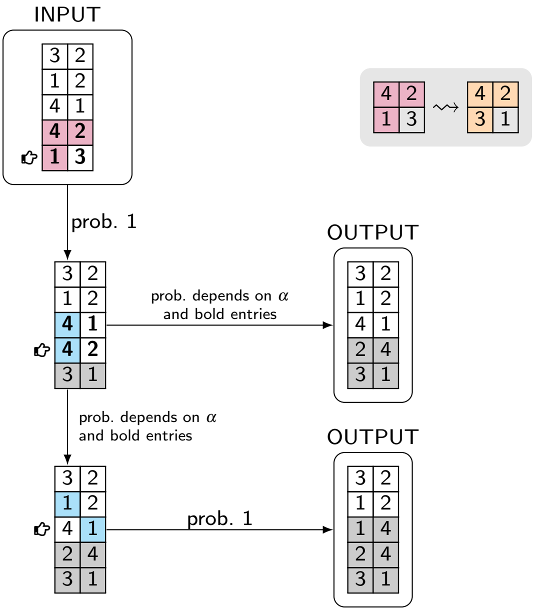
$\alpha = (3,4,4,0,0)$, $i = 2$

$\sigma = [5,1,3,4,2]$, $\sigma s_i = [5,3,1,4,2]$

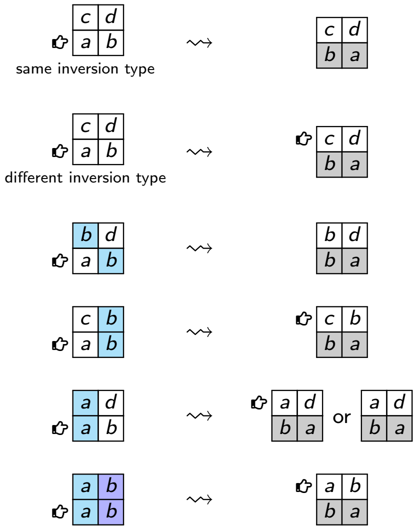
Input:

	3	**2**		
4	**1**	**2**		
3	**4**	**1**		
3	**4**	**2**		
5	**1**	**3**	4	2

Definition of the Probabilistic Operator by Example



All rules about moving the pointer ☛



It is a probabilistic bijection!

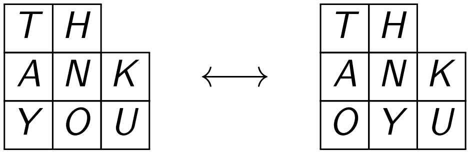
Theorem (D–Mandelshtam, 2025$^+$)

The algorithm describes a probabilistic bijection between $\mathrm{NAF}(\alpha,\sigma)$ *and* $\mathrm{NAF}(\alpha,\sigma s_i)$.

Summary

- Probabilistic bijections generalize weight-preserving bijections.
- Write the permuted basement Macdonald polynomial E_α^σ as the generating function of non-attacking fillings $\mathsf{NAF}(\alpha, \sigma)$.
- Construct a probabilistic bijection between $\mathsf{NAF}(\alpha, \sigma)$ and $\mathsf{NAF}(\alpha, \sigma s_i)$ when $\alpha_i = \alpha_{i+1}$.
- Conclude that, when $\alpha_i = \alpha_{i+1}$,

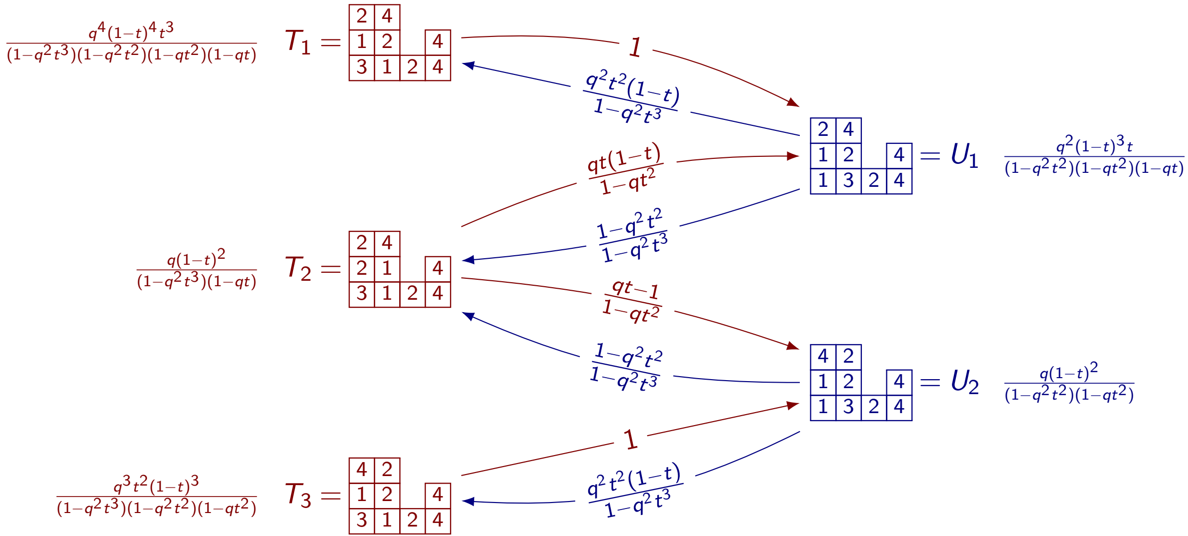
$$E_\alpha^\sigma = \sum_{T \in \mathsf{NAF}(\alpha,\sigma)} \mathrm{wt}(T) = \sum_{U \in \mathsf{NAF}(\alpha,\sigma s_i)} \mathrm{wt}(U) = E_\alpha^{\sigma s_i}.$$



Part IV

Appendix

$\alpha = (2,2,0,1)$, $\sigma = [3,1,2,4]$, content $(1,2,0,2)$



Future Work: $\alpha_i \neq \alpha_{i+1}$ via probabilistic bijections

If $\alpha_i < \alpha_{i+1}$ and $\sigma_i > \sigma_{i+1}$, then

$$E^{\sigma}_{s_i\alpha}(X;q;t) \quad = \quad E^{\sigma s_i}_{\alpha}(X;q;t) \;+\; \frac{t^{\phi(\alpha,\sigma,i)}(1-t)}{1-q^{L+1}t^{A}} E^{\sigma}_{\alpha}(X;q;t)$$

where $A = \mathrm{arm}(x)$ and $L = \mathrm{leg}(x)$ for $x = (i+1, \alpha_i + 1) \in \mathrm{dg}(\alpha)$.

Idea for Proof of Balance Condition

- Define the component of the weight contributed by the row r such that

$$\mathrm{wt}_{q,t}(T) = \prod_{\text{row } r} \mathrm{wt}_{q,t}{}^{(r)}(T).$$

- Define the probability that the pointer moves up in row r denoted by $\rho^{(r)}(T)$.
- Prove "balance condition" row-by-row:

$$\mathrm{wt}_{q,t}{}^{(r)}(T) \cdot \rho^{(r)}(T) = \mathrm{wt}_{q,t}{}^{(r)}(U) \cdot \rho^{(r)}(U) \qquad \text{or}$$
$$\mathrm{wt}_{q,t}{}^{(r)}(T) \cdot \left(1 - \rho^{(r)}(T)\right) = \mathrm{wt}_{q,t}{}^{(r)}(U) \cdot \left(1 - \rho^{(r)}(U)\right),$$

with the appropriate choice of ρ (move pointer up) or $1 - \rho$ (delete).

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